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The Space $\tilde{D}_k$ Land Weak Convergence for the Rectangle-Indexed Processes under Mixing

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In this paper we study the weak convergence of the weighted empirical processes indexed by rectangles of $[0,1]^k$ under both weak and strong mixing conditions. This is accomplished by generalizing the Skorohod topology on a space of functions defined on a set of rectangles.

1. INTRODUCTION

Let $X_{ni} = (X_{ni}^{(1)}, \ldots, X_{ni}^{(k)})$, $1 \le i \le n$, $n \ge 1$ be $\mathbb{R}^k$-valued random variables with continuous d.f.s (distribution functions) F_{ni} and continuous marginal d.f.s $F_{ni}^{(j)}$ of $X_{ni}^{(j)}$, $1 \le j \le k$, $1 \le i \le n$. Let $F_n^{(j)} = n^{-1}\sum_{i=1}^n F_{ni}^{(j)}$, and let $\{H_{ni}\}$ be a sequence of measures on $[0,1]^k$ defined by

$$H_{ni}(t_1, \ldots, t_k) = F_{ni}\left(F_n^{(1)^{-1}}(t_1), \ldots, F_n^{(k)^{-1}}(t_k)\right), \qquad 1 \le i \le n. \quad (1.1)$$

Let $\tilde{W}_n$ be an empirical process defined by

$$\tilde{W}_n(B) = n^{-1/2} \sum_{i=1}^{n} \left(\prod_{j=1}^{k} I_{[a_j < F_n^{(j)}(X_{ni}^{(j)}) \le b_j]} - H_{ni}(B) \right), \quad (1.2)$$

where $B = \prod_{j=1}^k (a_j, b_j] \subset [0,1]^k$ when $I_{[\ ]}$ denotes the indicator of [ ].

Our aim is to study the asymptotic behavior of $\tilde{W}_n$ for a certain Skorohod topology (to be defined in Section 3) when the sequence $\{X_{ni}\}$ is

$$\varphi\text{-mixing with rates } \varphi(m) = O(m^{-1-\varepsilon}) \quad (1.3)$$

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or

$$\text{strong mixing with rates } \sum_{m=1}^{\infty} m^{2(k+1)}\alpha^{\varepsilon}(m) < \infty$$

$$\text{for some } \varepsilon \in \left(0, \frac{1}{2(k+2)}\right). \quad (1.4)$$

Recall that $\{X_{ni}\}$ is φ-mixing if $\sup\{|P(B|A) - P(B)|; A \in \sigma(X_{ni}, 1 \le i \le j),\ B \in \sigma(X_{ni},\ i \ge j+m)\} = \varphi(m) \downarrow 0$ for positive j and m; and it is strong mixing if $\sup\{|P(A \cap B) - P(A) \cdot P(B)|; A \in \sigma(X_{ni}, 1 \le i \le j), B \in \sigma(X_{ni}, i \ge j+m)\} = \alpha(m) \downarrow 0$ for positive integers j and m. Here $\sigma(X_{ni}, 1 \le i \le j)$ and $\sigma(X_{ni}, i \ge j+m)$ are the σ-fields generated by $(X_{n1}, \ldots, X_{nj})$ and $(X_{n,j+m}, X_{n,j+m+1}, \ldots)$, respectively.

Bass and Pyke [4] extended the M_2 Skorohod [18] topology for functions on $[0,1]$ to set functions that are outer continuous with inner limits, but the empirical processes indexed by rectangles that we are considering have trajectories concentrated on the set of functions for which an extension of the Skorohod [18] J_1 topology is more appropriate. Most of the work dealing with the empirical processes indexed by points is concentrated on the Skorohod's J_1 topology. In their basic paper, Bass and Pyke [4] posed an open problem of determining a general class of sets on which an extension of Skorohod's J_1 topology is possible. In this paper we provide an extension of the Skorohod J_1 topology to functions indexed by rectangles and thus answer a part of the problem posed by Bass and Pyke [4]. We may also mention an interesting paper of Straf [19] which deals with the extension of J_1 topology to very general index sets. This requires the existence of a group Λ of homeomorphisms on an arbitrary space and does not seem to give real answers perhaps of its very theoretical nature. Also Straf's methods seem to be applicable on the space of functions defined only on $[0,1)^k$ and not on $[0,1]^k$, and they do not seem to be applicable to empirical processes indexed by rectangles.

The weak convergence of the weighted univariate empirical process indexed by points was established for the independent case by Pyke and Shorack [13], and for the φ-mixing case by Fears and Mehra [8] and later by Ahmad and Lin [1]. The generalization to the φ-mixing multivariate nonstationary case was carried on by Harel [9]. Shorack and Wellner [17] established the weak convergence of the weighted univariate empirical process indexed by intervals when the underlying random variables are independent. Their results were later generalized by Einmahl, Ruymgaart, and Wellner [7] to the multivariate case by using directly the well known Skorohod construction when the underlying random variables are independent, identical and uniformly distributed over $[0,1]^k$. Later Ruymgaart and Wellner [16] considered the case when the random variables are not uniformly distributed, but left open the problem of the convergence of the weighted empirical processes (see Ruymgaart and Wellner [16, remark, p. 221]. Our method of constructing Skorohod topology on the space of functions indexed by rectangles leads also to the extension of the results of Ruymgaart and Wellner [16] for more general classes of distributions (not necessarily uniform distributions).

We may also mention for reference the work of Alexander [2] on weighted empirical processes indexed by Vapnik–Cervonenkis classes of sets for the independent case. For the weak convergence of (nonweighted) empirical processes the reader is referred to the interesting papers of Neuhaus [10, 11] for the independent case, Rüschendorf [14] and Balacheff and Dupont [3] for the φ-mixing cases.

2. The $\tilde{D}_k$ and $\tilde{C}_k$ Spaces and Preliminaries

We write $t = (t_1, \dots, t_k)$, and half-open rectangles $R(t, t') = \prod_{j=1}^{k}(t_j, t_j']$. By convention, any point $t_j \in [0,1]$ will be called a half-open interval and will be written as $(t_j, t_j]$. Note that $R(t,t) = t$. For $(t,t') \in ([0,1]^k)^2$, $t \le t'$ will mean $t_j \le t_j'$ $\forall j = 1, \dots, k$, and $t < t'$ will mean $t_j < t_j$ $\forall j = 1, \dots, k$. For $(t,t') \in ([0,1]^k)^2$ with $t < t'$ or $t \le t'$ we can associate a rectangle $R(t,t')$ defined as before.

Let $\mathscr{I}(k) = \{R(t,t');\ R(t,t') \subset [0,1]^k\}$, and associate with the space $\mathscr{I}(k)$ the Hausdorff metric d_H, where $d_H(R(t,t'), R(s,s')) = \max_{1 \le j \le k} \max\{|s_j - t_j|, |s_j' - t_j'|\}$.

Consider a family of k strictly increasing finite sequences of elements of $[0,1]$ such that 0 is the first element of each sequence. For example, let $B = \{t_{ji}\}$, $1 \le i \le n_j$, $1 \le j \le k$ be k sequences such that $t_{j1} = 0$ and $t_{ji} < t_{ji+1}$ $\forall i \in \{1, \dots, n_j - 1\}$ and $j \in \{1, \dots, k\}$. Now associate with B a set $I^{(j)}(i,l)$ defined as

$$I^{(j)}(i,l) = \begin{cases} \left\{\left(t_j, t_j'\right] \subset [0,1],\ t_{ji} \le t_j < t_{ji+1},\ t_{jl} \le t_j' < t_{jl+1}\right\} \\ \qquad\qquad \text{if } 1 \le i, l \le n_j - 1 \\ \left\{\left(t_j, t_j'\right] \subset [0,1],\ t_{ji} \le t_j < t_{ji+1},\ t_{jl} \le t_j' \le 1\right\} \\ \qquad\qquad \text{if } 1 \le i \le n_j - 1,\ l = n_j \end{cases} \tag{2.1}$$

and

$$I^{(j)}(n_j, n_j) = \left\{\left(t_j, t_j'\right] \subset [0,1];\ t_{jn_j} \le t_j \le t_j' \le 1\right\}.$$

(Note that $I^{(j)}(i,l) = \phi$ if $t_{jl} \le t_{ji}$.)

Then a partition G of $\mathscr{I}(k)$ defined as

$$G = \left\{ \prod_{j=1}^{k} I^{(j)}(i_j, l_j), 1 \le i_j \le n_j, 1 \le l_j \le n_j, \text{ and } \prod_{j=1}^{k} I^{(j)}(i_j, l_j) \ne \phi \right\} \tag{2.2}$$

will be called a grid of $\mathscr{I}(k)$ with base B.

Let S^* be a finite subset of $\mathscr{I}(k)$, where $S^* = \{R(a^{(1)}, b^{(1)}), \dots, R(a^{(p)}, b^{(p)})\}$. Then the base generated by S^* is the smallest base of the grid B such that $\{a_j^{(1)}, \dots, a_j^{(p)}\} \cup \{b_j^{(1)}, \dots, b_j^{(p)}\} \subset \{t_{j1}, \dots, t_{jn_j}\}$ $\forall j = 1, \dots, k$.

Let $R(t, t') \in \mathscr{I}(k)$ and $(\rho, \varepsilon) \in \{0,1\}^k \times \{0,1\}^k$. Then the (ρ, ε) quadrant of $\mathscr{I}(k)$ with top $R(t, t')$ is a subset $Q(R(t, t'), \rho, \varepsilon)$ of $\mathscr{I}(k)$ defined as

$$Q(R(t, t'), \rho, \varepsilon) = \prod_{j=1}^{k} Q_j(t_j, t_j', \rho_j, \varepsilon_j),$$

where

$$Q_j(t_j, t_j', \rho_j, \varepsilon_j) = \begin{cases} \{(s_j, s_j'] \subset [0,1], s_j < t_j, s_j' < t_j'\} & \text{if } \rho_j = \varepsilon_j = 0 \\ \{(s_j, s_j'] \subset [0,1], s_j \ge t_j, s_j' < t_j'\} & \text{if } \rho_j = 1, \varepsilon_j = 0 \\ \{(s_j, s_j'] \subset [0,1], s_j < t_j, s_j' \ge t_j'\} & \text{if } \rho_j = 0, \varepsilon_j = 1 \\ \{(s_j, s_j'] \subset [0,1], s_j \ge t_j, s_j' \ge t_j'\} & \text{if } \rho_j = \varepsilon_j = 1. \end{cases}$$

Let $R(t, t') \in \mathscr{I}(k)$ and $(\rho, \varepsilon) \in \{0,1\}^k \times \{0,1\}^k$ such that $Q(R(t, t'), \rho, \varepsilon) \ne \phi$. Then, it is easy to see that if G is a grid, there exists an $S \in G$ such that $S \cap Q(R(t, t'), \rho, \varepsilon)$ is a neighborhood of $R(t, t')$ in $Q(R(t, t'), \rho, \varepsilon)$ in the topology induced on $Q(R(t, t'), \rho, \varepsilon)$ by the Hausdorff metric.

Let S be a nonempty subset of $\mathscr{I}(k)$. S is called a *pavement* of $\mathscr{I}(k)$ if S is of the form $S = \prod_{j=1}^{k} S_j$, where for any $1 \le j \le k$, $\exists (a_j^{(1)}, b_j^{(1)}, a_j^{(2)}, b_j^{(2)}) \in [0,1]^4$ such that $S_j = \{(t_j, t_j'], a_j^{(1)} \le t_j \le a_j^{(2)}, b_j^{(1)} \le t_j' \le b_j^{(2)}\}$.

For any $R(s, s') \in S$, we call the pair (J, L) the *indicator* of $R(s, s')$ into S if $(J, L) \subset \{1, \dots, k\} \times \{1, \dots, k\}$ is such that

$$s_j \in \{a_j^{(1)}, a_j^{(2)}\} \ \forall j \in J \quad \text{and} \quad s_j \in (a_j^{(1)}, a_j^{(2)}) \ \forall j \notin J$$

$$s_l' \in \{b_l^{(1)}, b_l^{(2)}\} \ \forall l \in L \quad \text{and} \quad s_l' \in (b_l^{(1)}, b_l^{(2)}) \ \forall l \notin L.$$

We say F, a subset of S, is the *face* of $R(s, s')$ in S with indicator (J, L) if F is of the form

$$F = \{R(u,u') \in S; u_j = s_j \ \forall j \in J \text{ and } u'_l = s'_l \ \forall l \in L\}.$$

Note that if $(J, L) = (\phi, \phi)$, then $F = S$, and if $(J, L) = \{1, \dots, k\} \times \{1, \dots, k\}$, then $F = R(s, s')$.

We say that f: $\mathscr{I}(k) \to \mathbb{R}$ admits a (ρ, ε) limit in $R(t, t')$ if and only if the restriction $f|Q(R(t, t'), \rho, \varepsilon)$ admits a limit in $R(t, t')$ with respect to the metric d_H and the usual metric on $\mathbb{R}$. We shall denote the (ρ, ε) limit of f in $R(t, t')$ by $f(R(t, t') + O(\rho, \varepsilon))$. If $(\rho, \varepsilon) = ((1, \dots, 1), (1, \dots, 1))$, then the (ρ, ε) limit of f is $f(R(t, t'))$.

Denote by $\tilde{D}_k$, the set of maps $f: \mathscr{I}(k) \to \mathbb{R}$ such that for any $R(t, t') \in \mathscr{I}(k)$ and any $(\rho, \varepsilon) \in \{0, 1\}^k \times \{0, 1\}^k$ for which $Q(R(t, t'), \rho, \varepsilon) \neq \phi$, f admits a (ρ, ε) limit in $R(t, t')$.

Finally denote by $\tilde{C}_k$, the set of maps $f: \mathscr{I}(k) \to \mathbb{R}$ which are continuous in d_H and the usual metric in $\mathbb{R}$.

2.1. *Properties of the Spaces $\tilde{D}_k$ and $\tilde{C}_k$*

Let $f \in \tilde{D}_k$ and $R(t, t') \in \mathscr{I}(k)$, and for any $(J, L) \subset \{1, \dots, k\} \times \{1, \dots, k\}$, set

$$H(f, R(t,t'), J, L) = \max\left\{ \begin{array}{c} |f(R(t,t') + O(\rho,\varepsilon)) - f(R(t,t') + O(\rho',\varepsilon'))|, \\ Q(R(t,t'),\rho,\varepsilon) \neq \phi, Q(R(t,t'),\rho',\varepsilon') \neq \phi, \\ \text{where } \forall j \in J,\ \rho_j = \rho'_j \text{ and } \forall l \in L,\ \varepsilon_l = \varepsilon'_l \end{array} \right\}. \quad (2.3)$$

Note that if $(J', L') \subset (J, L)$, then $H(f, R(t, t'), J, L) \leq H(f, R(t, t'), J', L')$ and if $(J, L) = \{1, \dots, k\} \times \{1, \dots, k\}$, then $H(f, R(t, t'), J, L) = 0$. If $(J, L) = (\phi, \phi)$, then we shall denote $H(f, R(t, t'), J, L)$ by $H(f, R(t, t'))$.

LEMMA 2.1. *Let $f \in \tilde{D}_k$, $R(t, t') \in \mathscr{I}(k)$, $(J, L) \subset \{1, \dots, k\} \times \{1, \dots, k\}$ and $\eta > 0$. Let $B(R(t, t'), \alpha)$ be an open ball with center $R(t, t')$ and radius α such that $\forall(\rho, \varepsilon) \in \{0, 1\}^k \times \{0, 1\}^k$, $\forall(s, s') \in B(R(t, t'), \alpha) \cap Q(R(t, t'), \rho, \varepsilon)$,*

$$|f(R(s,s')) - f(R(t,t') + O(\rho,\varepsilon))| < \eta. \quad (2.4)$$

Then, for any $R(u, u') \in B(R(t, t'), \alpha)$ for which $t_j = u_j$ $\forall j \in J$, $t_j \neq u_j$ $\forall j \notin J$; $t'_l = u'_l$ $\forall l \in L$ and $t'_l \neq u'_l$ $\forall l \notin L$, we have

$$H(f, R(u,u'), J, L) \leq 2\eta. \quad (2.5)$$

Proof. For any $(\rho, \varepsilon) \in \{0,1\}^k \times \{0,1\}^k$ such that $Q(R(u,u'), \rho, \varepsilon) \neq \phi$, we denote by $(\bar{\rho}, \bar{\varepsilon})$, an element of $\{0,1\}^k \times \{0,1\}^k$ defined as

$$\text{if } j \in J,\ \bar{\rho}_j = \rho_j \qquad\qquad \text{if } l \in L,\ \bar{\varepsilon}_l = \varepsilon_l$$

$$\text{if } j \notin J,\ \bar{\rho}_j = \begin{cases} 0 & \text{if } u_j < t_j \\ 1 & \text{if } u_j > t_j \end{cases} \qquad \text{if } l \notin L,\ \bar{\varepsilon}_l \begin{cases} 0 & \text{if } u'_l > t_l \\ 1 & \text{if } u'_j < t_l \end{cases}$$

then $Q(R(u,u'), \rho, \varepsilon) \cap Q(R(t,t'), \bar{\rho}, \bar{\varepsilon}) \neq \phi$.

Let $\eta_1 > 0$ and $\alpha' > 0$ be such that

$$|f(R(s,s')) - f(R(u,u') + O(\rho,\varepsilon))| < \eta_1$$

$\forall(\rho, \varepsilon) \in \{0,1\}^k \times \{0,1\}^k$ and $\forall R(s,s') \in B(R(u,u'), \alpha') \cap Q(R(u,u'), \rho, \varepsilon)$.

As $B(R(u,u'), \alpha') \cap Q((R(u,u'), \rho, \varepsilon) \cap B(R(t,t'), \alpha) \cap Q(R(t,t'), \bar{\rho}, \bar{\varepsilon}) \neq \phi$ we can find $R(s,s')$ such that (using (2.4))

$$|f(R(s,s')) - f(R(u,u') + O(\rho,\varepsilon))| < \eta_1$$

and

$$|f(R(s,s')) - f(R(t,t') + O(\bar{\rho},\bar{\varepsilon}))| < \eta.$$

From the above inequalities, we deduce

$$|f(R(t,t') + O(\bar{\rho},\bar{\varepsilon})) - f(R(u,u') + O(\rho,\varepsilon))| < \eta + \eta_1.$$

Since η_1 is arbitrary, the left side of the above inequality $\leq \eta$.

Let (ρ', ε') be another element of $\{0,1\}^k \times \{0,1\}^k$ such that $Q(R(u,u'), \rho', \varepsilon') \neq \phi$. Since $(\bar{\rho}', \bar{\varepsilon}') = (\bar{\rho}, \bar{\varepsilon})$, we deduce, proceeding analogously,

$$|f(R(u,u') + O(\rho',\varepsilon')) - f(R(t,t') + O(\bar{\rho},\bar{\varepsilon}))| \leq \eta$$

and

$$|f(R(u,u') + O(\rho',\varepsilon')) - f(R(u,u') + O(\rho,\varepsilon))| \leq 2\eta.$$

Now using (2.3), we conclude (2.5).

LEMMA 2.2. *Let $f \in \tilde{D}_k$ and $(J, L) \subset \{1, \ldots, k\} \times \{1, \ldots, k\}$. Set $T \subset \mathscr{I}(k)$ be such that $\forall(R(t,t'), R(s,s')) \in T \times T$ with $R(t,t') \neq R(s,s')$, we have $t_j = s_j$ $\forall j \in J, t_j \neq s_j$ $\forall j \notin J, t'_l = s'_l$ $\forall l \in L$, and $t'_l \neq s'_l$ $\forall l \notin L$. Then, for any $\eta > 0$, the set of elements of T such that $H(f, R(t,t'), J, L) \geq \eta$ is finite.*

Proof. Let $\eta > 0$ be fixed, and let T_η be the set of elements of T such that $H(R(t,t'), J, L) \geq \eta$. If T_η is infinite, then it has a limiting point

$R(t,t')$, and we can find a sequence $R(t^{(n)}, t'^{(n)})$ of points of T_η which admits $R(t,t')$ as limit, and is such that $\forall n \in \mathbb{N}, t_j^{(n)} \neq t_j$ for $\forall j \notin J$ and $t_l'^{(n)} \neq t_l'$ $\forall l \notin L$.

Let $\eta' < \eta/2$ and $\alpha > 0$ be such that $B(R(t,t'),\alpha)$ satisfies $|f(R(s,s')) - f(R(t,t'),\rho,\varepsilon)| < \eta'$ $\forall(\rho,\varepsilon) \in \{0,1\}^k \times \{0,1\}^k$ and $\forall R(s,s') \in B(R(t,t'),\alpha) \cap Q(R(t,t'),\rho,\varepsilon)$. Also let $n_0 \in \mathbb{N}$ be such that $R(t^{(n)}, t'^{(n)}) \in B(R(t,t'),\alpha)$ $\forall n \geq n_0$. Then, using Lemma 2.1, it follows that $H(f, R(t^{(n)}, t'^{(n)}), J, L) \leq 2\eta' < \eta$ which contradicts the hypothesis.

THEOREM 2.1. *Let R be the set of grids of $\mathscr{I}(k)$ and let $f: \mathscr{I}(k) \to \mathbb{R}$. Then $f \in \tilde{D}_k$ if and only if $\forall\eta > 0, \exists\delta > 0, \exists G \in R$ such that $\forall S \in G$ and $\forall(R(t,t'), R(s,s')) \in S \times S$ we have*

$$d_H(R(t,t'), R(s,s')) \leq \delta \Rightarrow |f(R(t,t') - f(R(s,s'))| < \eta. \quad (2.6)$$

Proof. (Sufficiency). Let $f: \mathscr{I}(k) \to \mathbb{R}$, and let $f \notin \tilde{D}_k$. Then we show that (2.4) is not true, that is we show that $\forall\delta > 0$, $\forall G \in R$, $\exists S \in G$, $\exists(R(t,t'), R(s,s') \in S \times S$ such that $d_H(R(t,t'), R(s,s')) \leq \delta$ and $|f(R(t,t') - f(R(s,s'))| \geq \eta$. Since $f \notin \tilde{D}_k$, it follows that $\exists R(u,u') \in \mathscr{I}(k)$ and $\exists(\rho,\varepsilon) \in \{0,1\}^k \times \{0,1\}^k$ such that $Q(R(u,u'),\rho,\varepsilon) \neq \phi$ and f has no (ρ,ε) limit in $R(u,u')$. This means that $\forall\eta_1 > 0$, $\exists(R(t,t'), R(s,s')) \in Q^2(R(u,u'),\rho,\varepsilon)$ such that $d_H(R(t,t'), R(s,s')) < \eta_1$ and $|f(R(t,t')) - f(R(s,s'))| \geq \eta$.

Let $\delta > 0$ and $G \in R$ be fixed. Then we can find $S \in G$ such that $Q(R(u,u'),\rho,\varepsilon) \cap S$ is a neighborhood of $R(u,u')$ in $Q(R(u,u'),\rho,\varepsilon)$. Then, there exists an $\alpha > 0$ such that

$$Q(R(u,u'),\rho,\varepsilon) \cap B(R(u,u'),\alpha) \subset Q(R(u,u'),\rho,\varepsilon) \cap S.$$

Now choose $\eta_1 = \min(\delta, \alpha/2)$. Then

$$\forall(R(t,t'), R(s,s')) \in (Q(R(u,u'),\rho,\varepsilon) \cap B(R(u,u'),\alpha))^2$$

such that $d_H(R(t,t'), R(s,s')) < \eta_1$. We see that $(R(t,t'), R(s,s')) \in S \times S$ and $d_H(R(t,t'), R(s,s') \leq \delta$.

We can now choose $(R(t,t'), R(s,s'))$, such that $|f(R(t,t') - f(R(s,s'))| \geq \eta$ and this leads to contradiction. Sufficiently is established.

(*Necessary part*). Let $f \in \tilde{D}_k$. Choose an $\eta > 0$, and let $\mathscr{H}$ be a class of subsets T of $\mathscr{I}(k)$ such that

(a) $\forall R(t,t') \in T$, $H(f, R(t,t')) \geq \eta$

(b) $\forall(R(t,t'), R(s,s')) \in T \times T$, $R(t,t') \neq R(s,s') \Rightarrow t_j \neq s_j$ $\forall j \in \{1,\ldots,k\}$ and $t_l' \neq s_l'$ $\forall l \in \{1,\ldots,k\}$.

Note that T is finite $\forall T \in \mathscr{H}$. Let S_1^* $(\in \mathscr{H})$ be a maximal element (in $\mathscr{H}$).

By iteration, we construct a sequence $S_1^*, \ldots, S_{2k}^*$ of sets in $\mathscr{I}(k)$ as follows: For any $R(s^{(1)}, s'^{(1)}) \in S_1^*$ and any $(J, L) \in \{1, \ldots, k\} \times \{1, \ldots, k\}$ such that Card J + Card $L = 1$, let $\mathscr{H}(R(s^{(1)}, s'^{(1)}), J, L)$ be a class of subsets T^* of $\mathscr{I}(k)$ such that

(c) $\forall R(t, t') \in T^*$, $H(f, R(t, t'), J, L) \geq \eta$

(d) $\forall R(t, t') \in T^*$, $t_j = s_j^{(1)}$ $\forall j \in J$ and $t_j' = s_j'^{(1)}$ $\forall l \in L$ and

(e) $\forall (R(t, t'), R(s, s')) \in T \times T$, $R(t, t') \neq R(s, s') \Rightarrow t_j \neq s_j$ $\forall j \notin J$ and $t_l' \neq s_l'$ $\forall l \notin L$.

Now let $S^*(R(s^{(1)}, s'^{(1)}), J, L)$ be a maximal element in $\mathscr{H}(R(s^{(1)}, s'^{(1)}), J, L)$. We set $S_2^* = S_1^* \cup_{(1)} \cup_{(2)} S^*(R(s^{(1)}, s'^{(1)}), J, L)$, where $\cup_{(1)}$ in the union over $R(s^{(1)}, s'^{(1)}) \in S_1^*$ and $\cup_{(2)}$ is the union over $(J, L) \in \mathscr{P}_1$, where $\mathscr{P}_1$ is the class of subsets of $J \subset \{1, \ldots, k\}$ and $L \subset \{1, \ldots, k\}$ such that Card J + Card $L = 1$.

Proceeding this way we get a sequence $S_1^* \subset \cdots \subset S_{2k}^*$ of sets in $\mathscr{I}(k)$. Let G be a grid generated by S_{2k}^*. Now denote

$$J(R(t, t')) = \{j \in \{1, \ldots, k\} : \exists R(s, s') \in S_{2k}^* \text{ and } s_j = t_j\}$$

and

$$L(R(t, t')) = \{l \in \{1, \ldots, k\} : \exists R(s, s') \in S_{2k}^* \text{ and } s_l' = t_l'\}.$$

Then, we first prove the following lemma.

Lemma 2.3. *Let $R(t, t') \in \mathscr{I}(k)$. Then for every (J, L) such that $(J(R(t, t'), L(R(t, t')) \subset (J, L)$, we have*

$$H(f, R(t, t'), J, L) < \eta. \tag{2.7}$$

Proof. Let $\mathscr{F}$ be a set of sequences $(R(s^{(1)}, s'^{(1)}), j_1), \ldots, (R(s^{(h)}, s'^{(h)}), j_h)$, where $1 \leq h \leq 2k$, $j_l \in \{1, \ldots, 2k\}$, and $1 \leq l \leq h$, such that,

$$s_{j_l}^{(l)} = t_{j_l} \quad \text{if } j_l \leq k,\ s_{j_l-k}'^{(l)} = t_{j_l-k}' \text{ if } j_l > k\ (1 \leq l \leq h)$$
$$R(s^{(1)}, s'^{(1)}) \in S_1^*, \qquad R(s^{(l)}, s'^{(l)}) \in S^*\big(R(s^{(l-1)}, s'^{(l-1)}), J_1, L_1\big) \tag{2.8}$$

where $J_1 = \cup_{p < l-1, j_p \leq k}\{j_p\}$, $L_1 = \cup_{p < l-1, j_p > k}\{j_p - k\}$, and $j_l \notin \{j_1, \ldots, j_{l-1}\}$. Note that $\mathscr{F}$ has at least one maximal sequence and it is easy to check that

$$\forall l \in \{1, \ldots, h\}, \qquad s_{j_l}^{(h)} = t_{j_l} \quad \text{if } j_l \leq k \text{ and } s_{j_l-k}'^{(h)} = t_{j_l-k}' \text{ if } j_l > k \tag{2.9}$$

Since $J_1 \subset J(R(t,t'))$ and $L_1 \subset L(R(t,t'))$, it suffices to prove that any maximal sequence satisfies

$$H(f, R(t,t'), J_2, L_2) < \eta,$$
$$\text{where } (J_2, L_2) = (J_1, L_1) \text{ with } h = l - 1. \quad (2.10)$$

There are three cases to be disposed of.

Case 1. Let $h = 2k$. Then from (2.9), $R(t,t') = R(s^{(2k)}, s'^{(2k)})$, $J(R(t,t')) = L(R(t,t')) = \{1,\dots,k\}$ and so

$$H(f, R(t,t'), J(R(t,t')), L(R(t,t'))) = 0\ (< \eta).$$

Case 2. Let $\mathscr{F} = \phi$. Then $H(f, R(t,t')) < \eta$, and also for any $(J,L) \subset \{1,\dots,k\} \times \{1,\dots,k\}$, $H(f, R(t,t'), J, L) < \eta$.

Case 3. Let $1 \le h < 2k$. Then, for any $R(s,s') \in S^*(R(s^{(h)}, s'^{(h)}), J_2, L_2)$ and $j \notin \{j_1,\dots,j_h\}$, we have $s_j \ne t_j$ if $j \le k$, $s'_{j-k} \ne t'_{j-k}$ if $j > k$. Now if $H(f, R(t,t'), J_2, L_2) \ge \eta$, then the sequence would not be maximal, and so we have the contraction. This proves the lemma.

Now let G be a grid generated by S^*_{2k}, and let $S \in G$.

Let $\bar{S}$ denote the closure of S under d_H. (Note that $\bar{S}$ is a pavement.) If $R(t,t') \in \bar{S}$, and (J,L) is the indicator of $R(t,t')$ in $\bar{S}$, then we have $H(f, R(t,t'), J, L) < \eta$.

Now we prove that for any $S \in G$ $\exists \delta > 0$ such that $\forall (R(t,t'), R(s,s')) \in S \times S$,

$$d_H(R(t,t'), R(s,s')) \le \eta \Rightarrow |f(R(t,t')) - f(R(s,s'))| < \eta. \quad (2.11)$$

Suppose (2.11) is not true. Then, we can find an $S \in G$ such that $\forall \delta > 0$, $\exists (R(t,t'), R(s,s') \in S \times S$, such that

$$d_H(R(t,t'), R(s,s')) \le \delta \quad \text{and} \quad |f(R(t,t')) - f(R(s,s'))| > \eta. \quad (2.12)$$

In this case, we can extract two sequences $R(t^{(n)}, t'^{(n)})$ and $R(s^{(n)}, s'^{(n)})$ in S such that they converge to the same limit $R(t,t')$ in $\bar{S}$; furthermore, $\exists (\rho, \varepsilon) \in \{0,1\}^k \times \{0,1\}^k$ and $(\rho', \varepsilon') \in \{0,1\}^k \times \{0,1\}^k$ with $R(t^{(n)}, t'^{(n)}) \in Q(R(t,t'), \rho, \varepsilon)$ and $R(s^{(n)}, s'^{(n)})$ in $Q(R(s,s'), \rho', \varepsilon')$ for any $n \ge 1$ such that $|f(R(t^{(n)}, t'^{(n)})) - f(R(s^{(n)}, s'^{(n)})| > \eta$.

As $f \in \tilde{D}_k$, $\lim_{n\to\infty} f(R(t^{(n)}, t'^{(n)})) = f(R(t,t') + O(\rho, \varepsilon))$ and $\lim_{n\to\infty} f(R(s^{(n)}, s'^{(n)})) = f(R(t,t') + O(\rho', \varepsilon'))$.

Consequently,

$$|f(R(t,t') + O(\rho,\varepsilon)) - f(R(t,t') + O(\rho', \varepsilon'))| \ge \eta. \quad (2.13)$$

Let (J, L) be the indicator of $R(t, t')$ in $\bar{S}$. If $(J, L) = (\phi, \phi)$, $J(R(t, t')) = \phi$, and $L(R(t, t')) = \phi$, then $\mathscr{F}$ is ϕ and so $H(f, R(t, t')) < \eta$, which is not compatible with (2.13).

If $(J, L) \neq (\phi, \phi)$, the $\rho_j = \rho_j'$ $\forall j \in J$, and $\varepsilon_l = \varepsilon_l'$ $\forall l \in L$, and so

$$\begin{aligned}
&|f(R(t,t') + O(\rho,\varepsilon)) - f(R(t,t') + O(\rho',\epsilon'))| \\
&\quad \leq H(f(R(t,t'),J,L) \\
&\quad \leq H(f(R(t,t'),J(R(t,t')),L(R(t,t')) < \eta
\end{aligned}$$

(from Lemma 2.3) and this is not compatible with (2.13). This proves the theorem.

DEFINITION (Balacheff and Dupont [3]). A grid G' is *finer* than a grid G if and only if $\forall S' \in G'$, $\exists S \in G$ such that $S' \subset S$.

Property of a Finer Grid. Given any $\delta > 0$ and any grid G, we can find another grid G' finer than G such that the diameter of each element S' of G' is less than or equal to δ.

For any grid G with base $B = \{\{t_{ji}\}, 1 \leq i \leq n_j, 1 \leq j \leq k\}$, we associate the number $m(G)$, called permeability $m(G)$ of G, defined as

$$m(G) = \inf_{1 \leq j \leq k} \inf_{1 \leq i \leq n_j} \{|t_{ji} - t_{ji+1}|, t_{ji} \neq t_{ji+1}\}, \qquad t_{jn_j+1} = 1$$

by convention. (Note that this concept of permeability is an extension of the similar concept given by Neuhaus [10] for a rather specialized situation connected with the space D_k.) Now denote

$$R_\eta = \{G; m(G) > \eta \text{ for any } \eta > 0\}. \tag{2.14}$$

COROLLARY 2.1. *Let* f: $\mathscr{I}(k) \to \mathbb{R}$. *Then*

(a) $f \in \tilde{D}_k$ *if and only if* $\forall \eta > 0$, $\exists$ *a grid* $G \in R$ *such that* $\forall S \in G$ *and* $\forall (R(t,t'), R(s,s')) \in S \times S$, *we have* $|f(R(t,t')) - f(R(s,s'))| < \eta$.

(b) $f \in \tilde{D}_k$ *if and only if* $\forall \eta > 0$, $\exists \delta > 0$ *and* $G \in R_\delta$ *such that* $\forall S \in G$ *and* $\forall (R(t, t'), R(s, s')) \in S \times S$, *we have* $|f(R(t, t')) - f(R(s, s'))| < \eta$.

Proof. (a) is a consequence of Theorem 2.1 and the property of the finer grid, and (b) is a consequence of (a).

For any function f: $\mathscr{I}(k) \to \mathbb{R}$, and any $\delta > 0$, we define the "modulus of continuity" $\omega(f, \delta)$ in $\tilde{D}_k$ and $\tilde{\omega}(f, \delta)$ in $\tilde{C}_k$ as

$$\omega'(f,\delta) = \inf_{G \in R_\delta} \max_{S \in G} \sup_{(R(t,t'), R(s,s')) \in S \times S} |f(R(t,t')) - f(R(s,s'))| \tag{2.15}$$

and

$$\tilde{\omega}(f,\delta) = \sup\{|f(R(t,t')) - f(R(s,s'))|;\ (R(t,t'),R(s,s')) \in \mathscr{I}(k) \times \mathscr{I}(k),\ d_H(R(t,t'),R(s,s')) \le \delta\}. \tag{2.16}$$

Note that a function $g: \delta \to \omega'(f,\delta)$, where $\delta \in (0,1]$ is nondecreasing.

COROLLARY 2.2. *Let* $f: \mathscr{I}(k) \to \mathbb{R}$. *Then*

(a) $f \in \tilde{D}_k$ *if and only if* $\lim_{\delta \to 0} \omega'(f,\delta) = 0$

(b) $f \in \tilde{C}_k$ *if and only if* $\lim_{\delta \to 0} \tilde{\omega}(f,\delta) = 0$.

Proof. (a) is a consequence of Corollary 2.1(b), and (b) follows by definition.

Note that for any bounded function $f: \mathscr{I}(k) \to \mathbb{R}$, and any $\delta \in (0, \frac{1}{2})$, $\omega'(f,\delta) \le \tilde{\omega}(f,2\delta)$.

3. SKOROHOD TOPOLOGY ON $\tilde{D}_k$

3.1. *Preliminaries*

In what follows, Λ denotes the space of maps $h: [0,1] \to [0,1]$ which are nondecreasing, continuous, and bijective. $\lambda^{(k)}$ denotes the space of maps $\lambda: [0,1]^k \to [0,1]^k$, where $\lambda(t_1,\dots,t_k) = (\lambda_1(t_1),\dots,\lambda_k(t_k))$, $\lambda_j \in \Lambda$, $1 \le j \le k$.

$I_{(k)}$ denotes the identity map on $[0,1]^k$,

$$|||\lambda||| = \max_{1 \le j \le k} \sup_{0 \le t_j < s_j \le 1} \left| \log \frac{\lambda_j(t_j) - \lambda_j(s_j)}{t_j - s_j} \right|$$

for any $R(t,t') \in \mathscr{I}(f)$, $\lambda(R(t,t'))$ denotes an element $R(s,s')$ of $\mathscr{I}(k)$ defined by $\lambda_j(t_j) = s_j$ and $\lambda_j(t_j') = s_j'$, $1 \le j \le k$. For any bounded maps $f,g: \mathscr{I}(k) \to \mathbb{R}$, we denote

$$d(f,g) = \inf_{\lambda \in \Lambda^{(k)}} \max\{\|f - g \circ \lambda\|, \|\lambda - I_{(k)}\|\};$$

$$d_0(f,g) = \inf_{\lambda \in \Lambda^{(k)}} \max\{\|f - g \circ \lambda\|, |||\lambda|||\},$$

where $\|f - g \circ \lambda\| = \sup_{R(t,t') \in \mathscr{I}(k)}\{|f(R(t,t')) - g \circ \lambda(R(t,t'))|\}$ and $\|\lambda - I_{(k)}\| = \sup_{t \in [0,1]^k}\{|\lambda(t) - I_{(k)}(t)|\}$. We shall call the topologies associated with d and d_0 as Skorohod and modified Skorohod topologies, respectively.

LEMMA 3.1. *Let f_n: $\tilde{D}_k \to \mathbb{R}$ be a sequence of maps such that $f_n \to f \in \tilde{D}_k$ in Skorohod topology. Let $R(t, t') \in \mathscr{I}(k)$ be such that the restriction of f to the face of $R(t, t')$ in $\mathscr{I}(k)$ is continuous in $R(t, t')$. Then $\lim_{n \to \infty} f_n(R(t, t')) = f(R(t, t'))$.*

Proof. Proof follows by using the inequality

$$\begin{aligned} &|f_n(R(t,t')) - f(R(t,t'))| \\ &\quad \le |f_n(R(t,t')) - f(\lambda^{-1}(R(t,t')))| \\ &\qquad + |f(\lambda^{-1}(R(t,t'))) - f(R(t,t'))|, \end{aligned}$$

where λ^{-1} is the inverse function of λ.

Remark 3.1. Following Billingsley [5] and Neuhaus [10], the following facts can easily be established:

(i) Skorohod topology as well as modified Skorohod topology implies uniform topology.

(ii) Uniform topolopgy is finer than the Skorohod topology.

(iii) The modified Skorohod topology is finer than the Skorohod topology.

(iv) The Skorohod topology and the modified Skorohod topology are equivalent in $\tilde{D}_k$.

(v) The space $(\tilde{D}_k, d)$ is separable.

(vi) The space $(\tilde{D}_k, d_0)$ is complete.

(vii) $\forall \delta > 0$, $\omega'(f, \delta)$ is upper semicontinuous in $f \in \tilde{D}_k$ with respect to the Skorohod topology.

LEMMA 3.2. *For any $R(t, t') \in \mathscr{I}(k)$, let $\varphi_{R(t,t')}$: $\tilde{D}_k \to \mathbb{R}$ be a map defined by $\varphi_{R(t,t')} = f(R(t, t'))$. If the restriction of the face of $R(t, t')$ into $\mathscr{I}(k)$ is continuous, then $\varphi_{R(t,t')}$ is continuous with respect to the Skorohod topology.*

Proof. Consequence of Lemma 3.1.

THEOREM 3.1. *Let $K \subset \tilde{D}_k$. Then the closure of $\bar{K}$ of K (with respect to the Skorohod topology) is compact if and only if*

$$\sup_{f \in K} \|f\| < \infty \tag{3.1}$$

and

$$\lim_{\delta \to \infty} \sup_{f \in K} \omega'(f, \delta) = 0. \tag{3.2}$$

Proof. Let $\bar{K}$ be compact. Then (3.1) and (3.2) follow from Remark 3.1(vii). We now prove the sufficiency part.

Let $K \subset \tilde{D}_k$, and let (3.1) and (3.2) hold. Since $(\tilde{D}_k, d_0)$ is complete, we have to show that $\forall \eta > 0$, $\exists$ a finite subset $K(\eta)$ of $\tilde{D}_k$ which is η-net in K with respect to d_0.

Choose an $\eta > 0$, and an integer m such that $\eta > 1/m$, and $\omega'(f, 1/m) < \eta$ $\forall f \in K$.

Let $\mathscr{L}_m = \{j/m;\ 0 \le j \le m\}$ and let $R(\mathscr{L}_m)$ be the set of grids G of $\mathscr{I}(k)$, with each grid having the base of the form $\{t_{ji},\ 1 \le i \le n_j,\ 1 \le j \le k\}$, where $t_{ji} \in \mathscr{L}_m$ $\forall i$ and $\forall j$.

Let H be an η-net set of $[-\sup\|f\|, \sup\|f\|]$ and let $K(\eta)$ be the space of step functions of the form $\sum_{S \in G} \alpha_S I_S$, where $G \in R(\mathscr{L}_m)$ and for any $S \in G$, α_S belongs to H. Then it is easy to check that $K(\eta)$ is 2η-net in K with respect to d, and hence $K(\eta)$ is η-net in K with respect to d_0.

4. Weak Convergence of Probability Measures in $\tilde{D}_k$

4.1. *Measurability on* $\tilde{D}_k$

For any $T \subset \mathscr{I}(k)$, let φ_T denote the projection of $\tilde{D}_k$ in $\mathbb{R}^T$, and let $\tilde{\mathscr{D}}_k$ be the Borel σ-field generated by the Skorohod topology in $\tilde{D}_k$. For any $(J, L) \subset \{1, \dots, k\} \times \{1, \dots, k\}$, let $F_{J,L} = \{R(t, t');\ t_j = 1\ \forall j \in J$ and $t'_l = 1\ \forall l \in L\}$. Then, we have

THEOREM 4.1. *$\tilde{\mathscr{D}}_k$ is the restriction to $\tilde{D}_k$ of the σ-field $\mathscr{B}^{\mathscr{I}(k)}$ on $\mathbb{R}^{\mathscr{I}(k)}$, where $\mathscr{B}$ is the usual Borel σ-field on $\mathbb{R}$.*

Proof. First we show that $\mathscr{B}^{\mathscr{I}(k)} \subset \tilde{\mathscr{D}}_k$. To prove this we have to show that $\forall R(t, t') \in \mathscr{I}(k)$, the map $\varphi_{R(t,t')}$: $\hat{D}_k \to \mathbb{R}$ is measurable. If $t = t' = (1, \dots, 1)$, then the measurability of $\varphi_{R(t,t')}$ follows as a consequence of Lemma 3.1. If t or $t' \neq (1, \dots, 1)$, then setting $J = \{j;\ j \in \{1, \dots, k\},\ t_j \neq 1\}$ and $L = \{l;\ l \in \{1, \dots, k\},\ t'_l \neq 1\}$, we notice that $L \subset J$. Then, from the continuity of the right with respect to t_j and t'_j in $R(t, t')$, it follows that $\varphi_{R(t,t')} = \lim_{\varepsilon \to 0} h_\varepsilon$, where

$$h_\varepsilon(f) = \frac{1}{\varepsilon^{(\mathrm{Card}\, J + \mathrm{Card}\, L)}}$$

$$\times \int_{\prod_{j \in J} [t_j, t_j + \varepsilon) \prod_{l \in L} [t'_l, t'_l + \varepsilon)} \big(f(R(u, u'))(du_j)_{j \in J} (du'_l)_{l \in L} \big)$$

and

$$\varepsilon < \inf\Bigl(\inf_{j\in J}(1-t_j),\ \inf_{l\in L}(1-t_l'),\ \inf_{l\in L}(t_l'-t_l)\Bigr),$$

where $u_j = 1$ if $j \notin J$, and $u_l' = 1$ if $l \notin L$.

We prove that h_ε is continuous with respect to the Skorohod topology. For any $f \in \tilde{D}_k$, denote

$$C(f) = \{R(s,s');\ R(s,s') \in \mathscr{I}(k) \text{ and the restriction of } f \text{ to the face of } R(s,s') \text{ into } \mathscr{I}(k) \text{ is continuous in } R(s,s')\}, \tag{4.1}$$

and the map

$$\Pi_{J,L}\colon \mathscr{I}(k) \to [0,1]^J \times [0,1]^L \tag{4.2}$$

defined by

$$\Pi_{J,L}(R(s,s')) = \bigl((s_j)_{j\in J}, (s_j')_{l\in L}\bigr).$$

If $f_n \to f_0$ in the Skorohod topology, then $f_n(R(s,s')) \to f_0(R(S,s'))$ $\forall R(s,s') \in C(f_0)$, and from Lemma 2.2 we deduce that $\Pi_{J,L}(C^*(f_0))$, where C^* denotes the complement of C, has the Lebesque measure 0 in $[0,1]^J \times [0,1]^L$. It follows (from the Lebesgue theorem) that $h_\varepsilon(f_n) \to h_\varepsilon(f_0)$, and so h_ε is continuous in the Skorohod topology. This implies that h_ε is measurable and hence $\varphi_{R(t,t')}$ is measurable. Thus $\mathscr{B}^{\mathscr{I}(k)} \subset \tilde{\mathscr{D}}_k$. Now we prove that $\tilde{\mathscr{D}}_k \subset \mathscr{B}^{\mathscr{I}(k)}$.

Let $T \subset \mathscr{I}(k)$, and suppose T is dense in $\mathscr{I}(k)$ and also for $(J,L) \subset \{1,\dots,k\} \times \{1,\dots,k\}$, $F_{J,L} \cap T$ is dense in $F_{J,L}$. Then, we prove that $\tilde{\mathscr{D}}_k$ is generated by $\{\varphi_{R(t,t')};\ R(t,t') \in T\}$.

Following Billingsley [5, p. 121], because d_0 is separable, it suffices to show that for any $f \in \tilde{\mathscr{D}}_k$ and $r > 0$, the open ball $B(f,r) = \{g \in \tilde{D}_k; d_0(f,g) < r\}$ belongs to the σ-field generated by $\{\varphi_{R(t,t')};\ R(t,t') \in T\}$. Take a sequence $\{R(t^{(n)}, t'^{(n)}\}$ of T, and suppose that it is dense in $\mathscr{I}(k)$, and the intersection of this sequence with $F_{J,L}$ is also dense in $F_{J,L}$. For any $\eta < r$ and any $N^* \in \mathbb{N}$, denote

$$\begin{aligned} A_{N^*}(\eta) = \bigl\{g \in \tilde{D}_k,\ &|||\lambda||| < r-\eta,\\ &\bigl|g\bigl(R(t^{(i)},t'^{(i)})\bigr) - f\bigl(\lambda\bigl(R(t^{(i)},t'^{(i)})\bigr)\bigr| < r-\eta,\\ &0 \le i \le N^*, \text{for some } \lambda \in \Lambda^{(k)}\bigr\}. \end{aligned} \tag{4.3}$$

Then, it follows that

$$A_{N^*}(\eta) = \varphi^{-1}_{\{R(t^{(0)}, t'^{(0)}), \dots, R(t^{(N^*)}, t'^{(N^*)})\}}(H_{N^*}(\eta)), \tag{4.4}$$

where

$$\begin{aligned} H_{N^*}(\eta) = \{(a_0, \dots, a_{N^*}) \in \mathbb{R}^{N^*+1};\ &|||\,\lambda\,||| < r - \eta \text{ and} \\ &|a_i - f(\lambda(R(t^{(i)}, t'^{(i)})))| < r - \eta, \\ &0 \le i \le N^* \text{ for some } \lambda \in \Lambda^{(k)}\} \end{aligned} \tag{4.5}$$

which implies that $A_{N^*}(\eta)$ belongs to the σ-field generated by $\{\varphi_{R(t,t')}; R(t,t') \in T\}$.

Next, we show that $B(f,r) = \bigcup_{\eta \in Q \cap (0,r)}(\bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta))$, where Q is the set of rational numbers.

It is clear that $B(f,r) \subset \bigcup_{\eta \in Q \cap (0,r)}(\bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta))$. It remains to show that for each $\eta \in Q \cap (0,r)$,

$$\bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta) \subset B(f,r). \tag{4.6}$$

Let $g \in \bigcap_{N^* \in \mathbb{N}} A_{N^*}(\eta)$. Choose N^* and $\lambda(N^*) \in \Lambda^{(k)}$ such that

$$|||\,\lambda^{(N^*)}\,||| < r - \eta$$

and

$$|g(R(t^{(i)}, t'^{(i)})) - f(\lambda^{(N^*)}(R(t^{(i)}, t'^{(i)})))| < r - \eta \quad \forall 0 \le i \le N^*.$$

Following Billingsley [5, p. 122], we can find a subsequence $\{\lambda^{(i_{N^*})}; N^* \in \mathbb{N}\}$ of $\{\lambda^{(N^*)}, N^* \in \mathbb{N}\}$ such that $\lambda^{(i_{N^*})} \to \lambda \in \Lambda^{(k)}$, and $|||\,\lambda\,||| < r - \eta$.

For any $R(t,t') \in T$, we can find an i such that $t_j \le t_j^{(i)}$ and $t'_j \le t_j'^{(i)}$, and then we have

$$|g(R(t^{(i)}, t'^{(i)})) - f(\lambda^{(i_{N^*})}(R(t^{(i)}, t'^{(i)})))| < r - \eta$$

$\forall N^* \ge i,\ i \in \{0, \dots, i_{N^*}\}$. It follows that there exists a $(\rho(R(t,t')), \varepsilon(R(t,t'))) = ((\rho_j)_{1 \le j \le k}, (\varepsilon_l)_{1 \le l \le k}) \in \{0,1\}^k \times \{0,1\}^k$ such that

$$t_j = 1 \Rightarrow \rho_j = 1\ \forall 1 \le j \le k \qquad \text{and} \qquad t'_l = 1 \Rightarrow \varepsilon_l = 1\ \forall 1 \le l \le k \tag{4.7}$$

$$Q(R(t,t'), \rho, \varepsilon) \neq \phi \tag{4.8}$$

and

$$|g(R(t,t')) - f(\lambda(R(t,t')) + O(\rho,\varepsilon))| \le r - \eta. \tag{4.9}$$

We now prove that $\|g - f \circ \lambda\| \leq r - \eta$. To prove this, suppose there exists $R(u, u') \in \mathscr{I}(k)$ such that $|g(R(u, u')) - f(\lambda(R(u, u'))| > r - \eta$. Then we can find an $R(t, t') \in T$ such that for any $(\rho, \varepsilon) \in \{0, 1\}^k \times \{0, 1\}^k$ such that (4.9) is not satisfied. This leads to contradiction. Thus $\|g - f \circ \lambda\| \leq r - \eta$. Now since $|||\lambda||| < r - \eta$, it follows that $g \in B(f, r)$. (4.6) holds.

COROLLARY 4.1. *Let T be denumerable and dense in $\mathscr{I}(k)$. Also let $F_{J,L} \cap T$ be dense in $F_{J,L}$ $\forall(J, L) \subset \{1, \dots, k\} \times \{1, \dots, k\}$. Let $\mathscr{D}_T \subset \tilde{\mathscr{D}}_k$, where $\mathscr{D}_T = \{\varphi_U^{-1}(H_U)$, U is a finite subset of T, and H_U a Borel-subset of $\mathbb{R}^U\}$, and φ_U is the projection of D_{k+1} on $\mathbb{R}^U$. If P and Q are probability measures on $\tilde{\mathscr{D}}_k$, and if $P = Q$ on $\mathscr{D}_T$, then $P = Q$ on $\tilde{\mathscr{D}}_k$.*

Proof. Consequence of Theorem 4.1.

For any probability measures on $(\tilde{D}_k, \tilde{\mathscr{D}}_k)$, denote

$$T_P = \big\{R(t, t');\, R(t, t') \in \mathscr{I}(k);\\ P\big(\{f;\, \varphi_{R(t,t')} \text{ is discontinuous on } f\}\big) = 0\big\}$$

$$T_P' = \{R(t, t');\, R(t, t') \in \mathscr{I}(k);\\ P(\{f; \text{the restriction of } f \text{ to the face of } R(t, t') \text{ into}\\ \mathscr{I}(k) \text{ is discontinuous in } R(t, t')\}) = 0\}.$$

Then, it is clear that $T_P' \subset T_P$.

THEOREM 4.2. *$T_P \cap F_{J,L}$ is dense in $F_{J,L}$ $\forall(J, L) \subset \{1, \dots, k\} \times \{1, \dots, k\}$.*

Proof. It suffices to prove that $T_P' \cap F_{J,L}$ is dense in $F_{J,L}$. To prove this it suffices to show that $T_P' \cap F_{\phi,\phi}$ is dense in $F_{\phi,\phi}$. For any $R(t, t') \in F_{\phi,\phi}$, let

$$J_{R(t,t')} = \{f;\, f \text{ is discontinuous in } R(t, t')\},$$

$$C = \big\{R(t, t');\, R(t, t') \in F_{\phi,\phi};\, P\big(J_{R(t,t')}\big) \neq 0\big\},$$

and

$$J_{\{R(t,t'),\eta\}} = \{f;\, H(f, R(t, t')) \geq \eta\}.$$

Obviously,

$$C = \bigcup_{l \geq 1} \bigcup_{m \geq 1} \big\{R(t, t') \in F_{\phi,\phi};\, P\big(J_{\{R(t,t'),1/m\}}\big) \geq 1/l\big\}.$$

Now using Lemma 2.2 and proceeding as in Billingsley [5, p. 124], we find that $\{R(t, t');\ R(t, t') \in F_{\phi,\phi};\ P(J_{\{R(t,t')\}}) \geq \eta_1\}$ has a Lebesque measure zero. This implies that C has a Lebesgue measure zero. Consequently $T'_P \cap F_{\phi,\phi}$ is dense in $F_{\phi,\phi}$.

THEOREM 4.3. *The sequence $\{P_n\}_{n \geq 1}$ of probability measures on $(\tilde{D}_k, \tilde{\mathscr{D}}_k)$ converges weakly to a probability measure P on $(\tilde{D}_k, \tilde{\mathscr{D}}_k)$ if and only if*

$$\varphi_U(P_n) \text{ converges weakly to } \varphi_U(P) \text{ for all finite subsets } U \text{ of } T_P; \tag{4.10}$$

$$\text{the sequence is tight.} \tag{4.11}$$

Proof. The necessary part is obvious. To prove the sufficiency part, note that since the sequence $\{P_n\}$ is tight, it is weakly relatively compact, we have to prove (cf. Billingsley [5, Theorem 2.3]) that any subsequence of $\{P_n\}$ which converges admits P as limit. For this we use the same line of argument as in Billingsley [5, Theorem 15.1], the fact that for any probability measure Q and any finite subset U of T_Q, φ_U is a.s. Q-continuous, and the Corollary 4.1.

COROLLARY 4.2. *Let $\{P_n\}_{n \geq 1}$ be a sequence of probability measures on $(\tilde{D}_k, \tilde{\mathscr{D}}_k)$ such that*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P_n(f; \omega'(f,\delta) \geq \varepsilon) = 0 \qquad \forall \varepsilon > 0 \tag{4.12}$$

and there exists a probability measure P on $(\tilde{D}_k, \tilde{\mathscr{D}}_k)$ such that

$$\varphi_U(P_n) \to \varphi_U(P) \text{ weakly for any finite subset } U \text{ of } T_P. \tag{4.13}$$

Then,

$$P_n \to P \text{ weakly with respect to the Skorohod topology.} \tag{4.14}$$

Proof. Follows from Theorems 3.1 and 4.3.

COROLLARY 4.3. *Let $\{P_n\}_{n \geq 1}$ be a sequence of probability measures on $(\tilde{D}_k, \tilde{\mathscr{D}}_k)$ such that*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P_n(f; \tilde{\omega}(f,\delta) \geq \varepsilon) = 0 \qquad \forall \varepsilon > 0 \tag{4.15}$$

and

$$(P_n) \text{ converges weakly to some probability measure } P_U \text{ on } R^U \text{ for every finite subset } U \text{ of } \mathscr{I}(k). \tag{4.16}$$

Then,

$$P_n \text{ converges weakly to some probability measure } P \text{ in Skorohod topology with } P(\tilde{C}_k) = 1. \tag{4.17}$$

Proof. Follows from Corollary 4.2 and the inequality $\omega'(f,\delta) \leq \tilde{\omega}(f,\delta)$.

5. Convergence of the Process $\tilde{W}_n$

5.1. *Preliminaries and Some Basic Tools*

In this section we study the asymptotic behavior of the empirical process $\tilde{W}_n$ defined in (1.2). Before we do that we introduce the space D_k. Let f: $[0,1]^k \to \mathbb{R}$. For $\rho \in \{0,1\}^k$, define

$$f_\rho(t) = \lim_{\substack{s_i \uparrow t_i,\, \rho(i)=1 \\ s_i \downarrow t_i,\, \rho(i)=0}} f(s)\big((s,t) \in [0,1]^k\big),$$

if it exists; in which case, call $f_\rho(t)$ the ρ-limit of f at t. Denote by D_k, the space of all maps f: $[0,1]^k \to \mathbb{R}$ such that for all $\rho \in \{0,1\}^k$, f_ρ exists and $f_\rho = f$ for $\rho = (0,\dots,0)$.

For any map f: $[0,1]^k \to \mathbb{R}$, and any rectangle $B = \prod_{j=1}^k (a_j, b_j]$, we denote a difference operator $\Delta_B f$ by

$$\Delta_B f = \Sigma(-1)^{\operatorname{card} I} f\big((b_i)_{i \in I}, (a_i)_{i \notin I}\big), \tag{5.1}$$

where $\operatorname{card} I$ is the cardinal of I, and Σ is over all the 2^k subsets $I \subset \{1,\dots,k\}$.

We will study the asymptotic behavior of $\tilde{W}_n$ via the empirical process W_n defined as

$$W_n(t) = n^{-1/2} \sum_{i=1}^n \left(\prod_{j=1}^k I_{[F_n^{(j)}(X_{ni}^{(j)}) \le t_j]} - H_{ni}(t) \right). \tag{5.2}$$

The process W_n has been studied by Balacheff and Dupont [3], among others. (See the references in Balacheff and Dupont [3].)

It is well known that the process W_n has a.s. (almost sure) trajectories in D_k. Set $B = \prod_{j=1}^k (a_j, b_j] \in \mathscr{I}(k)$, and by convention we put $\tilde{W}_n(B) = 0$ if there exists at least one j for which $b_j = a_j$. Then the process $\tilde{W}_n$ has a.s. trajectories in the space $\tilde{D}_k$.

Our results of Section 5 as well as Section 6 are based on the following two lemmas.

Lemma 5.1. *Let the sequence $\{X_{ni}\}$ of real-valued random variables centered at their expectations be φ-mixing with rates $\Sigma_{m \ge 1} m^{-1} \varphi^{1/2q}(m) < \infty$. Let N_n be the number of indexes i $(1 \le i \le n)$ for which X_{ni} is not identically zero. Set $S_n = \Sigma_{i=1}^n X_{ni}$, and $\|X_{ni}\|_l = (\int |X_{ni}|^{2l}\, dP_n)^{1/2l}$. Then, there exists a constant $C_q(\varphi)$ depending only on $q \in N^* = \{1,2,\dots\}$ and φ such that*

$$E(S_n^{2q}) \le C_q(\varphi) \sum_{l=1}^q N_n^{q/l} \Big(\sup_{1 \le i \le n} \|X_{ni}\|_l \Big)^{2q}. \tag{5.3}$$

Proof. The proof is a slight modification of Theorem 2.1 of Neumann [12] and is therefore omitted. See also Harel and Puri ([21], Lemma 4.1).

LEMMA 5.2. *Let the sequence $\{X_{ni}\}$ of real-valued random variables centered at their expectations be strong mixing with rates $\sum_{m>1} m^{2q-2}\alpha^{\varepsilon}(m) < \infty$, $q \geq 1$, $\varepsilon \in (0, 1/2q)$, and $|X_{ni}| \leq 1$, $1 \leq i \leq n$, $n \geq 1$. Let N_n be the number of indexes i $(1 \leq i \leq n)$ for which X_{ni} is not identically zero. Set $S_n = \sum_{i=1}^{n} X_{ni}$ and $\|X_{ni}\|_{\varepsilon} = (\int |X_{ni}|^{2/(1-\varepsilon)})^{1-\varepsilon}$. Then, there exists a constant $C_q(\alpha)$ depending only on q and α such that*

$$E\left(S_n^{2q}\right) \leq C_q(\alpha) \sum_{l=1}^{q} N^l \left(\sup_{1 \leq i \leq n} \|X_{ni}\|_{\varepsilon}^{l} \right). \tag{5.4}$$

Proof. The proof is essentially the same as in Doukhan and Portal [6] and is therefore omitted.

Now for any grid G with base $B = \{t_{ji}, 1 \leq i \leq n_j, 1 \leq j \leq k\}$, the number $\tau = \max_{1 \leq j \leq k} \max_{1 \leq j \leq n_j - 1}\{|t_{ji} - t_{ji+1}|\}$ is called the *pace* of the grid G.

Let $\{G_n\}_{n \geq 1}$ be a sequence of grids with paces $\{\tau_n\}_{n \geq 1}$. $\{G_n\}_{n \geq 1}$ is called asymptotically dense in $[0,1]^k$ if $\tau_n \to 0$ as $n \to \infty$. We denote the *lower boundary* of a subset C of $[0,1]^k$ by $\mathbf{C}$, where $\mathbf{C} = \{t;\ t \in C,\ t_j = 0$ for at least one j, $1 \leq j \leq k\}$. For any base $B = \{t_{ji}, 1 \leq i \leq n_j, 1 \leq j \leq k\}$, we say that a subset $\tilde{B}$ is tied to B if $\tilde{B} = \{t;\ t_j \in \{t_{j1}, \ldots, t_{jn_j}\}, 1 \leq j \leq k\}$ and $R(t, t')$ is *tied* to B if $t_j \in \{t_{j1}, \ldots, t_{jn_j}\}$ and $t'_j \in \{t_{j1}, \ldots, t_{jn_j}\}$ $\forall 1 \leq j \leq k$ and $|R(t, t')| \neq 0$, where $|R(t, t')|$ is the Lebesgue measure of $R(t, t')$, i.e., $|R(t, t')| = \prod_{j=1}^{k}(t'_j - t_j)$. Note that $\tilde{B}$ is unique whereas $R(t, t')$ is not unique.

For any base B of a grid G and the corresponding subset $\tilde{B}$ tied to B, denote

$$\omega_B(f, \delta) = \sup\{|f(t) - f(t')|;\ (t, t') \in \tilde{B} \times \tilde{B},\ \|t - t'\| \leq \delta\}$$
$$\text{for any } \delta > 0, \text{ where } \|t\| = \sup\left\{|t_j|, 1 \leq j \leq k\right\}. \tag{5.5}$$

Now for any bounded function $f\colon [0,1]^k \to \mathbb{R}$, denote

$$\omega(f, \delta) = \sup\left\{|f(t) - f(t')|, (t, t') \in [0,1]^k \times [0,1]^k, \|t - t'\| \leq \delta\right\}. \tag{5.6}$$

Let P_n, $n \geq 1$ be a sequence of probability measures on $(D_k, \mathscr{D}_k)$, where $\mathscr{D}_k$ is the σ-field generated by the Skorohod topology (on D_k). We say that the sequence $\{G_n\}$ of grids with bases $\{B_n\}$ *accompanies* the measure P_n if and only if $\forall \varepsilon > 0$, $\exists \varepsilon' > 0$ and $\forall \delta \in [0, 1/2)$, $\exists N_0 \geq 1$ such that $P_n[f \in D_k;\ \omega(f, \delta) \geq \varepsilon$ and $\omega_{B_n}(f, 2\delta) < \varepsilon'\}] = 0$ $\forall n \geq N_0$.

For the ease of convenience we state the following lemma due to Balacheff and Dupont [3] which will be used in the sequel.

LEMMA 5.3. *Let ν be a positive finite measure on $[0,1]^k$ with continuous marginals. Let $\{P_n\}_{n\geq 1}$ be a sequence of probability measures on $(D_k, \mathscr{D}_k)$ such that $P_n[f \in D_k;\ f|[0,1]^k = 0] = 1\ \forall n \geq 1$ where $f|[0,1]^k$ means the restriction of f on the lower boundary of $[0,1]^k$. Suppose $\{G_n\}_{n\geq 1}$, a sequence of grids with bases $\{B_n\}_{n\geq 1}$, is asymptotically dense in $[0,1]^k$, and accompanies P_n. Furthermore, suppose that for any $R(t^{(n)}, t'^{(n)}) \in \mathscr{I}(k)$ tied to B_n,*

$$P_n\left[f \in D_k;\ |\Delta_{R(t^{(n)}, t'^{(n)})} f| > \lambda\right] \leq \lambda^{-\gamma}\left[\nu\left(R(t^{(n)}, t'^{(n)})\right)\right]^{\beta},$$

$$\beta > 1, \gamma > 0. \quad (5.7)$$

Then, $\forall \varepsilon > 0$, $\exists \delta \in (0,1)$ and $N_0 \geq 1$ such that

$$P_n\left[f \in D_k;\ \omega(f, \delta) \geq \varepsilon\right] \leq \varepsilon\ \forall n \geq N_0. \quad (5.8)$$

Finally, we need the notion of μ-boundedness.

We shall say that the sequence $\{H_{ni}\}$ is μ-bounded if there exists a finite and positive measure μ on $[0,1]^k$ with continuous marginal distributions such that for every $1 \leq i \leq n$, $n \geq 1$, $H_{ni}(R(t,t')) \leq \mu(R(t,t'))$ for any rectangle $R(t,t')$ in $\mathscr{I}(k)$.

5.2. *Convergence of $\tilde{W}_n$*

THEOREM 5.1. *Assume that the sequence $\{X_{ni}\}$ is* (a) *φ-mixing with rates* (1.3) *or* (b) *strong mixing with rates* (1.4); *the sequence $\{H_{ni}\}$ is* (c) *μ-bounded, where μ is absolutely continuous with bounded density f_μ or* (d) *$\{H_{ni}\}$ has uniform marginals for all $n \geq 1$ and $1 \leq i \leq n$. Furthermore, assume that* (e) *the covariance function C_n of the empirical process $\tilde{W}_n$ defined in* (5.2) *converges to a function C. Then, $\tilde{W}_n$ converges weakly in the Skorohod topology to a Gaussian process $\tilde{W}_0$ with trajectories a.s. in $\tilde{C}_k$.*

Proof. Let the probability measure $\tilde{Q}_n$ (resp. Q_n) on $(\tilde{D}_k, \tilde{\mathscr{D}}_k)$ (resp. $D_k, \mathscr{D}_k)$) be associated with $\tilde{W}_n$ (resp. W_n). To prove this theorem, we have to verify (4.12) and (4.13). Then following Withers [20] $\varphi_U(\tilde{Q}_n) \to$ weakly to a Gaussian measure $\tilde{Q}_U$ if (i) $C_n \to$ some function C, (ii) $\Sigma_{m\geq 1}\alpha(m) < \infty$, and (iii) $m^{1-a}\alpha([m^b]) \to 0$ (as $m \to \infty$), where $0 < 2b < a < 1 - b$. Now in our situation (i) holds by assumption (e); (ii) follows from (1.3) and (1.4); and (iii) from (1.3) and (1.4) by taking $a = 3/4 - \varepsilon/8$, $b = 1/4$ and ε sufficiently small (since taking $\alpha(m) = m^{-1-\varepsilon}$, $m^{1-a}\alpha([m^b]) \leq Am^{-\varepsilon/8}$, where $A > 0$ is some constant). Thus (4.13) is proved.

We now prove (4.12). Since $|\tilde{W}_n(B)| = |\Delta_B W_n|$, where Δ_B is defined in (5.1), we have

$$\begin{aligned}\sup\{|\tilde{W}_n(R(t,t')) - \tilde{W}_n(R(s,s'))|;\, d_H(R(t,t'),R(s,s')) \le \delta\} \\ \le \sup\{2k|W_n(t) - W_n(s)|;\, \|t-s\| \le \delta\}. \qquad (5.9)\end{aligned}$$

To prove (4.12), it suffices to prove (5.8) for W_n. (5.8) will follow if we prove (5.7).

Let $B_n = \{i/n;\ 0 \le i \le n\}^k$, $n \ge 1$, be a sequence of bases of grids G_n, $n \ge 1$. Note that G_n is asymptotically dense in $[0,1]^k$, and we prove that G_n accompanies Q_n. Now for every $t \in [0,1]^k$, let $(\underline{t}, \bar{t})$ be the points of $\tilde{B}_n$, where $\tilde{B}_n$ is tied to B_n such that $\underline{t} < t \le \bar{t}$ and $\|\bar{t} - \underline{t}\| \le 1/n$. Then, with the conditions (c) or (d) we obtain, after some computations, that $|W_n(t) - W_n(t')| \le 2kK(\mu)/\sqrt{n} + |W_n(\bar{t}) - W_n(\underline{t}')|$ $\forall t \in [0,1]^k$ and $\forall t' \in [0,1]^k$, where $K(\mu) = \sup_{t\in[0,1]^k} f_\mu(t)$ if we have (c), and $K(\mu) = 1$ if we have (d). Consequently, for every $\delta \in (0, 1/2]$, we have $\omega(W_n, \delta) \le 2kK(\mu)/\sqrt{n} + \omega_B(W_n, 2\delta)$. It follows that G_n accompanies Q_n. It remains to show that Q_n satisfies (5.7).

Suppose we have condition (c). Let $\sum_{m=1}^\infty m^{-1}\varphi^{1/4}(m) < \infty$ (implied by (1.3)) and let $R(t^{(n)}, t'^{(n)})$ be tied to B_n. Using Lemma 5.1, with $q = 2$, we obtain

$$\begin{aligned}&E\big[\tilde{W}_n(R(t^{(n)}, t'^{(n)}))\big]^4 \\ &\quad \le C_2(\varphi)\left[\left(K(\mu)\prod_{j=1}^k (t_j' - t_j)\right)^2 + n^{-1}\left(K(\mu)\prod_{j=1}^k (t_j' - t_j)\right)\right]. \qquad (5.10)\end{aligned}$$

Let $\nu = (C_2(\varphi)(K(\mu) + K^2(\mu)))^{\beta^{-1}} U^k$, where U^k is the uniform probability measure on $[0,1]^k$ and $\beta = 1 + k^{-1}$. Then, by the Markov inequality, we obtain

$$Q_n\big[f \in D_k;\ |\Delta_{R(t^{(n)}, t'^{(n)})} f| > \lambda\big] \le \lambda^{-4}\big[\nu\big(R(t^{(n)}, t'^{(n)})\big)\big]^\beta$$

which implies (5.7) for the φ-mixing case with rates (1.3). Equation (5.8) follows.

For the strong mixing case with rates (1.4), we use Lemma 5.2 for $q = 2$ and $\varepsilon < (2k+4)^{-1}$ and obtain

$$\begin{aligned}&E\big[\tilde{W}_n(R(t^{(n)}, t'^{(n)}))\big]^4 \\ &\quad \le C_2(\alpha)\left[K(\mu)\prod_{j=1}^k (t_j - t_j')\right)^{2(1-\varepsilon)} + n^{-1}\left(K(\mu)\prod_{j=1}^k (t_j - t_j')\right)^{1-\varepsilon}\Bigg]\end{aligned}$$

which (with $\beta = (1-\varepsilon) + k^{-1}$) implies (5.7) and hence (5.8) by proceeding as above.

Now let us suppose we have condition (d). Then, for the φ-mixing case with rates $\sum_{m=1}^{\infty} m^{-1}\varphi^{1/2(k+1)}(m) < \infty$ (implied by (1.3)), we use Lemma 5.1 with $q = k+1$, and obtain

$$E\left[\tilde{W}_n\left(R\left(t^{(n)}, t'^{(n)}\right)\right)\right]^{2(k+1)} \le C_{k+1}(\varphi) \sum_{l=1}^{k+1} n^{-(k+1)} \left[n^{(k+1)/l} \prod_{j=1}^{k} \left(t_j' - t_j\right)^{(k+1)/kl} \right]$$

and proceeding as in the φ-mixing case dealt with above, we get the desired result.

For the strong mixing case with rates (1.4), we use Lemma 5.2 with $q = k+1$ and $\varepsilon < 1/2(k+2)$ and obtain

$$E\left[\tilde{W}_n\left(R\left(t^{(n)}, t'^{(n)}\right)\right)\right]^{2(k+1)} \le C_{k+1}(\alpha) \sum_{l=1}^{k+1} n^{-(k+1-l)} \prod_{j=1}^{k} \left(t_j' - t_j\right)^{l(1-\varepsilon)/k}$$

and then proceed as above for the first φ-mixing case. The proof follows.

6. Convergence of the Weighted Empirical Process

We start with the definition of the weight function.

Definition 6.1. A function r: $[0,1] \to \mathbb{R}^+$ is called a weight function if it satisfies the following conditions:

(i) r is continuous.

(ii) $r(u) = 0$ if $u = 0$ or $u = 1$.

We will consider a *modified empirical process* $\hat{W}_n$ defined as

$$\hat{W}_n(R(t,t)) = \begin{cases} \tilde{W}_n(R(t,t')) & \text{if } |R(t,t')| \ge n^{-1} \text{ and } |R(t,t')| \le 1 - n^{-1} \\ 0 & \text{otherwise}, \end{cases} \tag{6.1}$$

where $|R(t,t')|$ is defined in Section 5.

For any weight function r, we introduce a *weighted modified empirical process* $\hat{W}_n/r$ defined as

$$\frac{\hat{W}_n}{r}(R(t,t')) = \begin{cases} \hat{W}_n(R(t,t'))/r(|R(t,t')|) & \text{if } |R(t,t')| \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Then, the following lemma is a slight variation of a proposition in Harel [9].

LEMMA 6.1. *For any $n \geq 1$, let Y_n be a process with values in $\tilde{D}_k$, and measurable with respect to $\tilde{\mathscr{D}}_k$. Suppose $Y_n \to Y_0$ in law, where Y_0 is a Gaussian process with trajectories a.s. continuous. Let P_n be the probability measure associated with Y_n. Let r be a weight function such that*

$$Y_n \cdot 1/r \text{ has trajectories a.s. in } \tilde{D}_k, \ (n \geq 1) \tag{6.3}$$

$$\begin{aligned} &\forall \varepsilon > 0, \exists \theta > 0, \\ &\exists N_0 \geq 1 \text{ such that } P_n\big[\sup\{|Y_n(R(t,t'))(1/r)(|R(t,t')|)|\} \geq \varepsilon\big] \\ &\quad \leq \varepsilon \ \forall n \geq N_0, \end{aligned} \tag{6.4}$$

where sup *is over $R(t,t')$ with the condition that $|R(t,t')| \leq \theta$ or $1 - |R(t,t')| \leq \theta$. Then $Y_n \cdot 1/r$ converges weakly in Skorohod topology to the Gaussian process $Y_0 \cdot 1/r$ with trajectories a.s. in $\tilde{C}_k$.*

THEOREM 6.1. *If the sequence $\{X_{ni}\}$ satisfies the assumptions of Theorem 5.2, then for any weight function r satisfying*

$$r(u) \geq A[u(1-u)]^{1/2-\delta}, \qquad A > 0, \tag{6.5}$$

where

$$0 < \tfrac{1}{2} - \delta < \tfrac{1}{8}k \qquad \textit{if the condition } (\mathrm{c}) \textit{ of Theorem } 5.2 \textit{ is satisfied} \tag{6.6}$$

or

$$0 < \frac{1}{2} - \delta < \frac{1}{2k(k+1)}$$

$$\textit{if the condition } (\mathrm{d}) \textit{ of Theorem } 5.2 \textit{ is satisfied.} \tag{6.7}$$

$\hat{W}_n/r$ converges weakly in the Skorohod topology to the Gaussian process $\tilde{W}_0/r$ with trajectories a.s. in $\tilde{C}_k$ (where $\tilde{W}_0$ is the same as in Theorem 5.2).

Proof. Convergence of $\hat{W}_n$ to $\tilde{W}_0$ follows from the definition of $\hat{W}_n$ and Theorem 5.2. The theorem will follow if (6.3) and (6.4) are satisfied. Equation (6.3) follows from the definition of $\hat{W}_n$. We now prove (6.4).

For any $\theta \in [0, 1]$, let $C_\theta^{(1)}$ and $C_\theta^{(2)}$ be two subsets of $\mathscr{I}(k)$, where

$$C_\theta^{(1)} = \{R(t, t');\ |R(t, t')| \le \theta\}$$
$$C_\theta^{(2)} = \{R(t, t');\ |R(t, t')| \ge 1 - \theta\}.$$

Equation (6.4) will follow if we show that $\forall \eta > 0$, $\exists \theta > 0$, $\exists N_0 \ge 1$ such that

$$P_n\left[\sup_{R(t,t') \in C_\theta^{(i)}} \left| \hat{W}_n(R(t,t')) \cdot \frac{1}{r(|R(t,t')|)} \right| > \eta \right] \le \eta$$
$$\forall n \ge N_0,\ i = 1, 2. \quad (6.8)$$

First let us take $i = 1$. Without any loss of generality, let $r(u) = u^{1/2-\delta}$. Set (for any $n \ge 1$)

$$m_n = \max\left\{m;\ \frac{1}{n} \le \frac{1}{2^m},\ m \ge 0\right\} \quad (6.9)$$

and

$$m(\theta) = \max\{m;\ \theta \le 1/2^m,\ m \ge 0\}. \quad (6.10)$$

For any $p \ge 0$, consider a base B_p of some grid G_p, where

$$B_p = \left\{t_{ji};\ 1 \le i \le n_j^{(p)},\ 1 \le j \le k,\ t_{ji} = i/2^p,\ n_j^{(p)} = 2^p\right\}. \quad (6.11)$$

We need a few lemmas.

Lemma 6.1. *Let $\theta \in [0, 1]$ and suppose a function $f\colon [0, 1] \to \mathbb{R}$ is given. Then, for any two points*

$$(u, v) \in \left\{0, \frac{1}{2^m}, \frac{2}{2^m}, \ldots, \frac{2^m - 1}{2^m}, 1\right\} \times \left\{0, \frac{1}{2^m}, \frac{2}{2^m}, \ldots, \frac{2^m - 1}{2^m}, 1\right\}$$

with $|u - v| \le 2^{-m(\theta)}$, where m is an integer $> m(\theta)$, the inequality

$$|f(u) - f(v)| \le 4 \sum_{r=m(\theta)}^{m} \sup\left|f\left(u_1 + 2^{-4}\right) - f(u_1)\right|,$$

holds, where the sup *is taken for all*

$$u_1 \in \left\{0, \frac{1}{2^r}, \frac{2}{2^r}, \ldots, \frac{2^r - 1}{2^r}, 1\right\} \quad \text{and} \quad u_1 + 2^{-r} \in [0, 1].$$

Proof. Follows from Neuhaus [10, Lemma 5.1].

LEMMA 6.2. *Let $\theta \in [0,1]$ and suppose a function f: $[0,1]^k \to \mathbb{R}$ be given. Then, for any $(t,t') \in B_m \times B_m$ with $t < t'$ and $|R(t,t')| \le 2^{-m(\theta)}$, the inequality*

$$\frac{|\Delta_{R(t,t')}f|}{r(|R(t,t')|)} \le 4^k \sum_{\substack{r_1 \le m \\ r_1 + \cdots + r_k \ge m(\theta)}} \cdots \sum_{r_k \le m} \sup\left\{ \frac{|\Delta_{\prod_{j=1}(u_j, u_j + 2^{-r_j}]}f|}{\left(\prod_{j=1}^{k} 2^{-r_j}\right)^{1/2-\delta}} \right\} \tag{6.13}$$

holds, where the sup *is taken over all*

$$u_j \in \left\{0, \frac{1}{2^{r_j}}, \ldots, \frac{2^{r_j}-1}{2^{r_j}}\right\} \quad \text{and} \quad u_j + 2^{-r_j} \in [0,1], 1 \le j \le k,$$

and where m is an integer $> m(\theta)$.

Proof. From Lemma 6.1, we deduce (by iteration)

$$|\Delta_{R(t,t')}f| \le 4^k \sum_{\substack{r_1 \le m \\ r_1 + \cdots + r_k \ge m(\theta)}} \cdots \sum_{r_k \le m} \sup\{|\Delta_{\prod_{j=1}^k (u_j, u_j + 2^{-r_j}]}f| . \tag{6.14}$$

Since each term on the left side of (6.14) is less than or equal to a finite number of terms on the right side of (6.14) for which the Lebesgue measure $|R(t,t')| \ge \prod_{j=1}^k 2^{-r_j}$ and, since r is a nondecreasing function, we obtain (6.13).

LEMMA 6.3. $\forall \eta > 0, \exists \theta > 0, \exists N \ge 1$ *such that*

$$P_n\left[\sup_{\substack{R(t,t') \in \tilde{B}_{m_n} \cap C_\theta^{(1)} \\ |R(t,t')| \neq 0}} \left\{ \left| \tilde{W}_n(R(t,t')) \cdot \frac{1}{r(|R(t,t')|)} \right| \ge \eta \right\} \right] \le \eta \ \forall n \ge N. \tag{6.15}$$

Proof. See Appendix.

Now we prove (6.8) for $i = 1$. We assume that condition (c) of Theorem 5.2 holds. Let $\theta \in [0,1]$ and n be fixed. For any $R(t,t') \in \mathscr{I}(k)$ for which $|R(t,t')| \ge n^{-1}$, let $\underline{t}, \bar{t}, t'$, and $\bar{t}'$ be points of $\tilde{B}_{m_n}$ such that

$$\underline{t}_j \le t_j \le \bar{t}_j, \underline{t}_j' \le t_j' \le \bar{t}_j', \bar{t}_j - \underline{t}_j \le 2^{-m_n} \quad \text{and} \quad \bar{t}_j' - \underline{t}_j' \le 2^{-m_n},$$
$$1 \le j \le k.$$

For any $(J, L) \subset \{1, \ldots, k\} \times \{1, \ldots, k\}$ for which $J \cap L = \phi$, we define an element $R(t, t')(J, L)$ of $\mathscr{I}(k)$ by

$$R(t,t')(J,L) = \prod_{j \in J} (\underline{t}_j, t_j] \prod_{j \in L} (t_j', \bar{t}_j'] \prod_{j \notin J \cup L} (\underline{t}_j, \bar{t}_j'].$$

Then, we have the inequality:

$$\left| \hat{W}_n(R(t,t')) \cdot \frac{1}{r(|R(t,t')|)} \right| \leq \sum_{\substack{(J,L) \subset \{1,\ldots,k\} \times \{1,\ldots,k\} \\ J \cap L = \phi}} \left| \tilde{W}_n(R(t,t')(J,L)) \frac{1}{r(|R(t,t')|)} \right|. \quad (6.16)$$

If $J = L = \phi$, then we have the inequality

$$\left| \tilde{W}_n(R(t,t')(\phi,\phi)) \frac{1}{r(|R(t,t')|)} \right| \leq 3^k \left| \tilde{W}_n(R(\underline{t},\bar{t}')) \frac{1}{r(|R(\underline{t},\bar{t}')|)} \right|. \quad (6.17)$$

If $J \cup L \neq \phi$, then we have the inequality

$$\left| \tilde{W}_n(R(t,t')(J,L)) \frac{1}{r(|R(t,t')|)} \right| \leq \tilde{W}_n(G_k) \times \frac{1}{r(|R(t,t')|)} \Bigg| + n^{1/2} K(\mu) U^k(G_k) \cdot \frac{1}{r(|R(t,t')|)}, \quad (6.18)$$

where

$$G_k = \prod_{j \in J} (\underline{t}_j, \bar{t}_j] \prod_{j \in L} (\underline{t}_j', \bar{t}_j'] \prod_{j \notin J \cup L} (\underline{t}_j, \bar{t}_j'].$$

Equation (6.18) follows due to the fact that for any $(B_1, B_2) \in \mathscr{I}(k) \times \mathscr{I}(k)$, where $B_2 \subset B_1$,

$$|\tilde{W}_n(B_2)| \leq |\tilde{W}_n(B_1)| + n^{1/2} \mu(B_1). \quad (6.19)$$

As r is non-decreasing, we deduce

$$\left|\tilde{W}_n(R(t,t')(J,L))\frac{1}{r(|R(t,t')|)}\right|$$
$$\leq 3^k|\tilde{W}_n(G_k)| \times \left(r(|G_k|)\right)^{-1}$$
$$+ n^{1/2}\left(K(\mu)\prod_{j\in J}(\bar{t}_j - \underline{t}_j)\prod_{j\in L}(\bar{t}_j - \underline{t}'_j)\right)^{(1/2+\delta)}\}. \quad (6.20)$$

Now

$$n^{1/2}\left(\prod_{j\in J}(\bar{t}_j - \underline{t}_j)\prod_{j\in L}(\bar{t}'_j - \underline{t}'_j)\right)^{(\delta+1/2)} = n^{1/2}(2^{-m_n})^{(\delta+1/2)l} \to 0 \quad (6.21)$$

as $n \to \infty$, where $l = \text{Card } J \cup L$.

Using (6.16), (6.17), (6.20), and (6.21), we obtain

$$\sup_{R(t,t')\in C_\theta^{(1)}}\left|\hat{W}_n(R(t,t'))\cdot\frac{1}{r(|R(t,t')|)}\right|$$
$$\leq K \sup_{\substack{R(t,t')\in\tilde{B}_m\cap C_\theta^{(1)}\\ |R(t,t')|\neq 0}}\left|\tilde{W}_n(R(t,t'))\cdot\frac{1}{r(|R(t,t')|)}\right|$$
$$+ O\left(n^{1/2}(2^{-m_n})^{(\delta+1/2)l}\right), \quad (6.22)$$

where $K > 0$ is some constant.

Using (6.22) along with Lemma 6.3, we obtain (6.8) for $i = 1$ when condition (c) of Theorem 5.2 is satisfied. The proof when condition (d) of Theorem 5.2 is satisfied is essentially similar and is therefore omitted.

Now we prove (6.8) for $i = 2$. Without loss of generality, we take $r(u) = (1-u)^{1/2-\delta}$. Following the ideas of Einmahl, Ruymgaart, and Wellner [7], we start with the equality $\tilde{W}_n(B) = -\tilde{W}_n(B^*)$, where B^* is the complement of B and $\tilde{W}_n(B^*) = n^{-1/2}\sum_{i=1}^n[I_{[F_n(X_{ni})\in B^*]} - H_{ni}(B^*)]$, where $F_n = (F_n^{(1)}, \dots, F_n^{(k)})$. For any $R(t,t') \in \mathscr{I}(k)$ with $|R(t,t')| \leq 1 - n^{-1}$ and using the union-intersection principle (see Einmahl, Ruymgaart, and Wellner [7]), we obtain

$$|\tilde{W}_n(R(t,t'))| \leq \sum_{l\in L}|\tilde{W}_n(R_l(t,t'))|, \quad (6.23)$$

where L is a finite index set, $R_l(t,t')$ is an element of $\mathscr{I}(k)$ with the condition that there exists $J \subset \{1,\dots,k\}$ and $M \subset \{1,\dots,k\}$ with

$J \cap M = \phi$ and $J \cup M \neq \phi$, such that,

$$R_l(t,t') = \prod_{j \in J}(0,t_j] \cdot \prod_{j \in M}(t_j',1] \cdot \prod_{j \notin J \cup M}(0,1].$$

From (6.23), we obtain

$$\sup_{R(t,t') \in C_\theta^{(2)}} \left| \hat{W}_n(R(t,t')) \frac{1}{r(|R(t,t')|)} \right|$$
$$\leq \sum_{l \in L} \left| \tilde{W}_n(R_l(t,t')) \frac{1}{\left(|R_l(t,t')| \vee n^{-1}\right)^{1/2-\delta}} \right| \tag{6.24}$$

where $a \vee b = \max(a,b)$.

If $|R_l(t,t')| \geq n^{-1}$, then $R_l(t,t') \in C_\theta^{(1)}$ and from (6.8) for $i = 1$, we obtain the desired result for $\tilde{W}_n(R_l(t,t')) \cdot (r(|R(t,t')|))^{-1}$.

If $R_l(t,t')| < n^{-1}$, then using the inequality (6.19), we obtain

$$\left| \tilde{W}_n(R_l(t,t')) \cdot \frac{1}{\left(n^{-1}\right)^{1/2-\delta}} \right| \leq \left| \tilde{W}_n\left(R_l^{(n)}(t^{(n)},t'^{(n)})\right) \frac{1}{\left(n^{-1}\right)^{1/2-\delta}} \right|$$
$$+ n^{1/2}\mu\left(R_l^{(n)}(t^{(n)},t'^{(n)})\right)/\left(n^{-1}\right)^{1/2-\delta}, \tag{6.25}$$

where $R_l^{(n)}(t^{(n)},t'^{(n)}) \in \mathscr{I}(k)$, $|R_l^{(n)}(t^{(n)},t'^{(n)})| = n^{-1}$ and $R_l(t,t') \subset R_l^{(n)}(t^{(n)},t'^{(n)})$.

Since $R_l^{(n)}(t^{(n)},t'^{(n)}) \in C_\theta^{(1)}$, we obtain the desired result from (6.8) for $i = 1$ for $\tilde{W}_n(R_l^{(n)}(t^{(n)},t'^{(n)})) \cdot 1/r(|R_l^{(n)}(t^{(n)},t'^{(n)})|)$. Since $|R_l^{(n)}(t,t')| = n^{-1}$,

$$n^{1/2} \frac{\mu\left(R_l^{(n)}(t^{(n)},t'^{(n)}\right)}{\left(n^{-1}\right)^{1/2-\delta}} \leq K(\mu) \frac{n^{1/2} \cdot n^{-1}}{\left(n^{-1}\right)^{1/2-\delta}} = K(\mu) n^{-\delta} \to 0$$

as $n \to \infty$.

Using (6.24), and the properties of $\tilde{W}_n(R_l(t,t')) \cdot 1/(|R_l(t,t')| \upsilon n^{-1})^{1/2-\delta}$ for each $l \in L$ obtained in the discussion following (6.24), we obtain the desired result, viz. (6.8) for $i = 2$. The proof follows.

Appendix

Proof of Lemma 6.3. (a) Let us suppose that $\{X_{ni}\}$ is φ-mixing with rates (1.3) and $\{H_{ni}\}$ is μ-bounded (viz. conditions (a) and (c) of Theorem 5.1).

From Lemma 6.2, we obtain if $m(\theta) \le m_n$ that

$$\sup_{\substack{R(t,t') \in \tilde{B}_{m_n} \cap C_\theta^{(1)} \\ |R(t,t')| \ne 0}} \left| \tilde{W}_n(R(t,t')) \cdot \frac{1}{r(|R(t,t')|)} \right|$$

$$\le 4^k \sum_{\substack{r_1 \le m_n \\ r_1 + \cdots + r_k \ge m(\theta)}} \cdots \sum_{r_k \le m_n} \sup \left| \tilde{W}_n\left(\prod_{j=1}^k \left(u_j, u_j + 2^{-r_j}\right]\right) \right.$$

$$\left. \times \frac{1}{\left(\prod_{j=1}^k 2^{-r_j} \right)^{1/2-\delta}} \right|, \tag{A.1}$$

where the sup is taken as in (6.13). Let a be a real number such that

$$0 < a < 1 \qquad \text{and} \qquad a^4 \cdot 2^{1/k - 4(1/2-\delta)} > 1. \tag{A.2}$$

If the left side of (A.1) exceeds η, then $\exists (r_1, \ldots, r_k) \in \{1, \ldots, m_n\} \times \cdots \times \{1, \ldots, m_n\}$ with $r_1 + \cdots + r_k \ge m(\theta)$ such that

$$\sup \left\{ \left| \tilde{W}_n(A_k) \cdot \frac{1}{\left(\prod_{j=1}^k 2^{-r_j} \right)^{1/2-\delta}} \right| \right\} > B_k, \tag{A.3}$$

where

$$A_k = \prod_{j=1}^k \left(u_j, u_j + 2^{-r_j}\right], \qquad B_k = \frac{a^{\sum_{i=1}^k r_i - m(\theta)}}{4^k B_k^{(1)}} \eta$$

and

$$B_k^{(1)} = \sum_{r_1 + \cdots + r_k \ge m(\theta)} a^{\sum_{i=1}^k r_i - m(\theta)} = O\left(m^{k-1}(\theta)\right).$$

From (A.3), we deduce

$$P_n\left[\sup_{\substack{|R(t,t')|\neq 0\\ R(t,t')\in B_{m_n}\cap C_\theta^{(1)}}}\left\{\left|\tilde{W}_n(R(t,t'))\frac{1}{r(|R(t,t')|)}\right|\geq\eta\right\}\right]$$

$$\leq \sum_{\substack{r_1\leq m_n\\ r_1+\cdots+r_k\geq m(\theta)}}\cdots\sum_{r_k\leq m_n}\sum_{u_1=0}^{2^{r_1}-1}\cdots\sum_{u_k=0}^{2^{r_k}-1}$$

$$\times P_n\left[\left|\tilde{W}_n(A_k)\frac{1}{\left(\prod_{j=1}^{k}2^{-r_j}\right)^{1/2-\delta}}\right|>B_k\right]$$

$$\leq \sum_{\substack{r_1\leq m_n\\ r_1+\cdots+r_k\geq m(\theta)}}\cdots\sum_{r_k\leq m_n}\sum_{u_1=0}^{2^{r_1}-1}\cdots\sum_{u_k=0}^{2^{r_k}-1}(B_k^{-4})$$

$$\times E\left[\tilde{W}_n(A_k)\left(\prod_{j=1}^{k}2^{-r_j}\right)^{\delta-1/2}\right]^4 \tag{A.4}$$

by the Markov inequality.

Using Lemma 5.1 for $q = 2$, the right side of (A.4) is less than or equal to

$$\sum_{\substack{r_1\leq m_n\\ r_1+\cdots+r_k\geq m(\theta)}}\cdots\sum_{r_k\leq m_n}K_1(B_k)^{-4}\left(2^{-\sum_{j=1}^{k}r_j}\right)^{1/k-4(1/2-\delta)},$$

$K_1>0$ a constant,

$$\leq\left(\frac{4^k}{\eta}B_k^{(1)}\right)^4\sum_{\substack{r_1\leq m_n\\ r_1+\cdots+r_k\geq m(\theta)}}\cdots\sum_{r_k\leq m_n}b^{-m(\theta)}\cdot(ba^4)^{m(\theta)-\sum_{j=1}^{k}r_j},$$

$$b=2^{1/k-4(1/2-\delta)}>1,$$

$$\leq K_1\left(\frac{4^k}{\eta}B_k^{(1)}\right)^4B_k^{(2)}b^{-m(\theta)},$$

$$\text{where } B_k^{(2)}=\sum_{r_1+\cdots+r_k\geq m(\theta)}(ba^4)^{-\sum_{i=1}^{k}r_i+m(\theta)}. \tag{A.5}$$

Using (A.2), we note that $B_k^{(2)} = O(m^{k-1}(\theta))$ and, since $(m^{k-1}(\theta))^5\cdot b^{-m}(\theta)\to\infty$ as $\theta\to 0$ ($m(\theta)\to\infty$ as $\theta\to 0$), it follows that the extreme side of (A.5) $\to 0$ as $\theta \to 0$, which implies that $n \to \infty$. This proves (6.15) with conditions (a) and (c) of Theorem 5.2.1.

(b) Let $\{X_{ni}\}$ be strong mixing with rates (1.4), and let $\{H_{ni}\}$ be μ-bounded (viz. conditions (b) and (c) of Theorem 5.1). Choose a real number a such that

$$0 < a < 1 \qquad \text{and} \qquad a^4 \cdot 2^{1/k-\varepsilon-4(1/2-\delta)} > 1. \tag{A.6}$$

Now using Lemma 5.2 for $q = 2$, and proceeding as above, we obtain

$$P_n\left[\sup_{\substack{R(t,t')\in\tilde{B}_{m_n}\cap C_\theta^{(1)}\\ |R(t,t')|\neq 0}}\left|\tilde{W}_n(R(t,t')\cdot\frac{1}{r(|R(t,t')|)}\right|\geq\eta\right]$$

$$\leq K_2\left(\frac{4^k}{\eta}B_k^{(1)}\right)^4 B_k^{(2)}b^{-m(\theta)}, \tag{A.7}$$

where $b = 2^{1/k-\varepsilon-4(1/2-\delta)}$ and $K_2 > 0$ is a constant. Using (A.7) and arguing as in (A.5), we find that the right side of (A.7) $\to 0$ as $\theta \to 0$ (and therefore $n \to \infty$). This proves (6.15) with conditions (b) and (c) of Theorem 5.1.

(c) Let $\{X_{ni}\}$ be φ-mixing with rates (1.3) and let $\{H_{ni}\}$ have uniform marginals (viz. conditions (a) and (d) of Theorem 5.1).

Choose a real number a such that

$$1 < a < 1 \qquad \text{and} \qquad a^{2(k+1)}2^{1/k-2(k+1)(1/2-\delta)} > 1. \tag{A.8}$$

(This is always possible since from (6.7), $1/k - 2(k+1)(\frac{1}{2}-\delta) > 0$, and so we choose $a \in (0,1)$ such that (A.8) is satisfied.) Using Lemma 5.1 for $q = k+1$ and proceeding as above, we note that the left side of (A.7) $\leq K_3((4^k/\eta)B_k^{(1)})^{2k+2}B_k^{(3)}b^{-m(\theta)}$, where

$$B_k^{(3)} = \sum_{r_1+\cdots+r_k\geq m(\theta)} \left(ba^{2k+2}\right)^{-\sum_{i=1}^k r_i+m(\theta)} = O\left(m^{k-1}(\theta)\right),$$

$$b = 2^{1/k-2(k+1)(1/2-\delta)}$$

and $K_3 > 0$ is some constant. Using (A.8) and arguing as in (A.5), we prove (6.15) with conditions (a) and (d) of Theorem 5.1.

(d) Finally, let $\{X_{ni}\}$ be strong mixing with rates (1.4) and let $\{H_{ni}\}$ have uniform marginals (viz. conditions (b) and (d) of Theorem 5.1). Here choosing $0 < a < 1$ and $a^{2(k+2)}2^{(2-\varepsilon(k+1))k^{-1}-2(k+2)(1/2-\delta)} > 1$ using Lemma 5.2 for $q = k+2$, and proceeding as in (b), we get (6.15). This proves Lemma 6.3.

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